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INDUCED QCD FROM THE NONCOMMUTATIVE GEOMETRY OF A SUPERMANIFOLD

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We study the noncommutative geometry of a two-leaf Parisi–Sourlas supermanifold in Connes' formalism using different K-cycles over the Grassmann algebra valued functions on the supermanifold. We find that the curvature of the trivial noncommutative vector bundle defines in the simplest case the super Yang–Mills action coupled to a scalar field. By considering a modified Dirac operator and a suitable limit of its parameters we then obtain an action that turns out to be the continuum limit of the induced QCD in Kazakov–Migdal model.

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1 Introduction

Connes' theory of noncommutative geometry (NCG) provides us with a machinery to study the geometry of spaces, whose structure may be very different from that of the ordinary smooth manifolds [1]. It is believed that NCG might give new ideas to the geometrical study of quantum mechanics and the quantum field theories. Until now one has, however, worked mainly on trying to explain some features of classical field theories, as we will also do in this paper. As the result of these studies one now has *e.g.* a geometrical interpretation for the Higgs field of the Standard Model as a kind of a connection field into a discrete direction: the universe is supposed in a sense to be an unconnected manifold. An important result is also the fact that the Standard Model can be expressed in the terms of NCG [2]. Some grand unified theories [3], Chern–Simons theories [4] and Einstein–Hilbert gravity coupled to a scalar field [5] have their equivalents as the nontrivial geometry of some manifold in the sense of NCG. Also supersymmetric theories can be discussed in this setting [6], but problems in the definition of the integral of differential forms arise. These problems are present also in our work, when we study a supersymmetric manifold.

In this paper we first briefly review the basic concepts of Connes' NCG, and how classical field theory of scalar, spinor and vector fields emerges as a natural consequence of the geometry and the symmetries of some given manifold [4, 7, 10]. The emphasis is put on how to do the practical calculations. We also consider Feynman's path integral quantization as a means to anticipate the possible future implementation of quantum field theories in NCG. Next we take an example of a noncommutative manifold, which can best be described as a two-leaf supermanifold (Parisi–Sourlas type). We review briefly the structure of the Clifford algebra on this kind of a manifold [8, 9], and then apply the machinery of the NCG in order to study Yang–Mills theories on the manifold. The procedure here is similar to that first presented in [3]. Connes' theory does leave some freedom to choose the exact structure of the Yang–Mills action, and some of the possibilities to proceed are examined. The result is in general a super Yang–Mills theory coupled to a scalar field. Using in a certain way the freedom that we have in the choice of the geometry, we consider also a purely bosonic theory that one obtains after integration over Grassmann coordinates. In a particular limit of the parameters we obtain the continuum limit action of the Kazakov–Migdal model.

2 Basic concepts

The basic data that define a noncommutative geometry [1, 7] are a C^*-algebra $\mathcal{A}$ and a triple $(\mathcal{H}, \mathcal{D}, \Gamma)$ called a K-cycle. $\mathcal{H}$ is a Γ-graded Hilbert space and the operator $\mathcal{D}$, often called the Dirac operator, is an odd operator on $\mathcal{H}$. Elements of $\mathcal{A}$ are suitable even operators on $\mathcal{H}$.

For example, the geometry of an ordinary, *'commutative'*, manifold $\mathcal{M}$ is obtained, if the algebra $\mathcal{A}$ is the algebra $C^\infty(\mathcal{M}, \mathbf{C})$ of smooth, complex valued functions of the manifold, the Hilbert space $\mathcal{A}$ is the space of spinors on this manifold, the grading

is the Clifford grading, and the Dirac operator is the usual spin connection $\not{\partial} + \not{\omega}$. The studied manifold can be *'noncommutative'* in the sense that the algebra $\mathcal{A}$ is not commutative: An example of this is the case where the algebra above is tensored with the space of complex $n \times n$-matrices $\mathrm{Mat}(n, \mathbf{C})$, and the K-cycle is suitably modified. We will study a special case of this this later in detail.

The basic tool for the study of geometry is the differential algebra $\Omega^* \mathcal{A}$, whose homogenous elements

$$a^0 \mathrm{d}a^1 \cdots \mathrm{d}a^p \in \Omega^p \mathcal{A} \tag{1}$$

are defined as elements of the tensor algebra

$$\mathcal{A} \otimes \mathcal{A}/\mathbf{C} \otimes \cdots \otimes \mathcal{A}/\mathbf{C}. \tag{2}$$

Here two elements a and a' of the algebra $\mathcal{A}$ represent the same element in the quotient algebra $\mathcal{A}/\mathbf{C}$ if

$$a - a' = \lambda \mathbf{1}, \qquad \mathbf{1} \in \mathcal{A} \tag{3}$$

is valid for some complex number λ .

This differential algebra can be represented as operators on $\mathcal{H}$ using the homomorphism

$$\pi(a^0 \mathrm{d}a^1 \cdots \mathrm{d}a^p) := a^0 i[\mathcal{D}, a^1] \cdots i[\mathcal{D}, a^p]. \tag{4}$$

In the case of a superalgebra, i.e. when the elements of $\mathcal{A}$ are not necessarily even, one should use a different representation [10]. Representation π has the defect that the fact $\pi(\alpha) = 0$ for some differential form α does not imply $\pi(\mathrm{d}\alpha) = 0$. Because we need a *differential* representation of $\Omega^* \mathcal{A}$ we can thus not simply use $\pi(\Omega^* \mathcal{A})$. A suitable representation turns out to be the quotient algebra

$$\Omega^*_{\mathcal{D}} \mathcal{A} := \pi(\Omega^* \mathcal{A}) / \pi(\mathrm{d} \ker \pi). \tag{5}$$

The representation mapping is denoted by

$$\pi_{\mathcal{D}} \colon \Omega^* \mathcal{A} \to \Omega^*_{\mathcal{D}} \mathcal{A}. \tag{6}$$

Connes defines the integration of differential forms $\alpha \in \Omega^* \mathcal{A}$ using the Dixmier trace by the formula

$$\int \alpha = \mathrm{Tr}_\omega \, \pi_{\mathcal{D}}(\alpha) |\mathcal{D}|^{-n}, \tag{7}$$

where ω denotes a limiting procedure that picks up the logarithmic divergence of the conventional trace. The integer n is characteristic to the chosen K-cycle and is associated with the dimension of the studied manifold by analog of the commutative case. Once this integer is found, one says that the K-cycle is *n-summable*. In the case of a superalgebra that is not n-summable for any finite n this definition can not be applied. Instead, one can try heat kernel regularization [3, 10] and define

$$\int \alpha = \lim_{\beta \to 0+} \mathrm{Str} \, \pi_{\mathcal{D}}(\alpha) \; e^{-\beta \mathcal{D}^2}, \tag{8}$$

where Str is the usual functional trace of the Hilbert space. In this article, the algebra $\mathcal{A}$ is given using a matrix algebra of functions on a *a priori* know supermanifold. We can thus integrate differential forms using the formula

$$\int \alpha := \int \mathrm{d}^n x \; \mathrm{d}^2\eta \sqrt{|\operatorname{sdet} g\,|} \operatorname{Tr} \pi_{\mathcal{D}}(\alpha(x,\eta)), \tag{9}$$

where Tr is the ordinary, finite dimensional matrix trace. Other notation will be explained later.

The concept of a fiber bundle can also be introduced. In the following, only the trivial bundle $\mathcal{E} := \mathcal{A}$ is considered. On this generalization of a fiber bundle a covariant derivative is given by $\nabla = \mathrm{d} + \alpha, \;\alpha \in \Omega^1\mathcal{A}$. The antihermitian connection one-form α defines the curvature two-form $\vartheta := \nabla^2 = \mathrm{d}\alpha + \alpha^2$.

3 Field theory in NCG

Once a fiber bundle structure has been discovered, the natural thing to do is to start to study the associated Yang–Mills theory [1, 7] which, in general, is defined by the action

$$\mathcal{S}_{YM}[\alpha] := \int \vartheta^2 \tag{10}$$

α being the connection 1-form. This expression is not well defined as it stands, because the integration rules of the last section involve the representation $\pi_{\mathcal{D}}$ and the way of choosing a representative of an element in the quotient algebra is not yet given. This ambiguity can be solved in two, equivalent ways as it will turn out: either directly fix a way of projecting any abstract element of the differential algebra to its unique representation operator [4], or define the action as the *minimum* of all possible choices of the representative [7]. The first possibility amounts to defining an orthonormal basis in the space of differential forms with respect to the natural inner product of $\Omega^*\mathcal{A}$

$$\langle \alpha|\beta\rangle \;\; = \;\; \int \pi(\alpha\beta^*). \tag{11}$$

Using the notation $\alpha = \alpha^{\perp} + \alpha^{\|}$ where $\langle \alpha^{\perp}|\beta^{\|}\rangle = 0$ for all $\beta^{\|} \in d\ker\pi$ one then redefines

$$\mathcal{S}_{YM}[\alpha] = \int (\vartheta^{\perp})^2. \tag{12}$$

The second possibility, which we will use here, amounts to redefining the action in the following way: Define first the *'preaction'*

$$\mathcal{S}[\alpha] := \int \pi(\mathrm{d}\alpha + \alpha^2)^2. \tag{13}$$

This is equivalent to modifying the integration rules listed above such that instead of $\pi_{\mathcal{D}}$ there appears π. Now define

$$\mathcal{S}_{YM}[\alpha] := \inf_{\alpha' \in \Omega^1\mathcal{A}} \Big\{ \mathcal{S}[\alpha'] \;\Big|\; \pi(\alpha - \alpha') = 0 \Big\}. \tag{14}$$

Let us write the connection 1-form with the help of elements of $\mathcal{A}$ in form

$$\alpha = \sum_{n,m} a^n \mathrm{d}b^m, \qquad a^n, b^m \in \mathcal{A}. \tag{15}$$

Both $\pi(\alpha)$ and $\pi(\mathrm{d}\alpha + \alpha^2)$ are some functionals of the possibly very large set of fields (a^n, b^m). It is useful to attempt to write down $\pi(\mathrm{d}\alpha + \alpha^2)$ and the preaction with the help of $\pi(\alpha)$ and some functionally independent auxiliary field P. The Yang–Mills action is then the preaction evaluated for the fields $(\pi(\alpha), P^0)$, where the fields P^0 is the solution of the functional equation

$$\left.\frac{\delta S[\pi(\alpha), P]}{\delta P}\right|_{P=P^0} = 0. \tag{16}$$

If the preaction turns out to be of the form $\mathcal{S}[\alpha] = \int \mathrm{Tr}(\mathcal{L}[\pi(\alpha)] + P)^2$ as it will in our later application, the Euler–Lagrange equation 16 just implies that the effective lagrangian is the projection of $\mathcal{L}[\pi(\alpha)]$ to the orthogonal direction with respect to the space of auxiliary fields P. This establishes the equivalence of the two methods. Moreover, we notice that the Yang–Mills action thus depends only on rather $\pi(\alpha)$ than α.

Fermions can be introduced to the theory as vectors of $\mathcal{H}$. A suitable action is

$$\mathcal{S}_F[\Psi, \alpha] = \langle \Psi | \mathcal{D} - i\pi(\alpha) | \Psi \rangle, \tag{17}$$

where $\langle \cdot | \cdot \rangle$ is a scalar product in $\mathcal{H}$.

Both actions $\mathcal{S}_{YM}$ and $\mathcal{S}_F$ are invariant under unitary transformations of $\mathcal{E}$. These transformations turn out to be usual local gauge transformations. Under such a transformation

$$u \in \mathrm{U}(\mathcal{E}) = \mathrm{U}(\mathcal{A}) = \{ v \in \mathcal{A} \mid v^* v = v v^* = \mathbf{1} \} \tag{18}$$

the transformation rules for the objects introduced before are

$$\begin{aligned} \gamma_u(\Psi) &= u\Psi & (19)\\ \gamma_u(\alpha) &= u\alpha u^* + u\mathrm{d}u^* & (20)\\ \gamma_u(\vartheta) &= u\vartheta u^*. & (21) \end{aligned}$$

It will turn out to be useful to find the transformation rule of the connection form in terms of elements of $\mathcal{A}$. Let

$$\alpha = \sum_{n,m} a^n \mathrm{d}b^m, \qquad a^n, b^m \in \mathcal{A}. \tag{22}$$

Here the summation runs over a suitable set of elements of $\mathcal{A}$ that does not necessarily need to be any kind of a basis of the product algebra $\mathcal{A} \otimes \mathcal{A}/\mathbf{C}$. The index set can be large, since for example on a circle S^1 we can use the decomposition

$$a^n \mathrm{d}b^m = e^{inx} \otimes e^{imy}; \qquad n, m \in \mathbf{Z}, m \neq 0; \quad x, y \in [0, 2\pi[. \tag{23}$$

In the following, only one index is used to enumerate elements of $\Omega\mathcal{A}$ for convenience. Adding to any α given in this form the term $(\Lambda - \sum a^n \mathrm{d}b^n)\mathrm{d}\mathbf{1}$ for any $\Lambda \in \mathcal{A}$ does not change its value in $\Omega^*\mathcal{A}$, but now the new coefficients a^n, b^n satisfy

$$\sum_n a^n b^n = \Lambda. \tag{24}$$

In the case $\Lambda = \mathbf{1}$ transformation properties for the constituents of α reduce to

$$\begin{aligned} \gamma_u(a^n) &= ua^n && (25)\\ \gamma_u(b^n) &= b^n u^*. && (26) \end{aligned}$$

By now, there is no established way of formulating quantum field theories in NCG. However, since we do know how to obtain classical field theories, we can always quantize them in Feynman's path integral formalism. Because the Yang–Mills and the fermion action only depend on $\pi(\alpha)$ rather than α, as was previously shown, one should attempt to define the generating functional by

$$[\mathrm{d}\Psi]\ \exp\Big(-\mathcal{S}_{YM}[\alpha] \int \pi(\alpha)^* \mathcal{J}_F\Big), \mathcal{W} = \int [\mathrm{d}\pi(\alpha)][\mathrm{d}\Psi]\ \exp\Big(-\mathcal{S}_{YM}[\alpha] - \mathcal{S}_F[\alpha, \Psi]\Big). \tag{27}$$

are some elements of When the connection one form is normalized such that $\Lambda = \mathbf{1}$ in formula 24 one can, as was shown, treat the $\mathrm{U}(\mathcal{E})$ invariance of the original action as the local gauge invariance of the spinors Ψ. The gauge field is naturally $\pi(\alpha)$. The way we choose to represent the algebra $\mathcal{A}$ in Hilbert space $\mathcal{H}$ thus fixes the gauge group. One has to remove this invariance of the integrand by inserting a gauge condition and a Faddeev–Popov determinant. However, this naive quantization method does not profit from any of the various means the NCG would provide for better understanding of quantum field theories. The purpose of the treatment above is merely to demonstrate the significance of the representation $\pi(\alpha)$ compared to the abstract 1-form α.

4 The two-leaf supermanifold $\mathcal{M}$

The rest of the paper is devoted for studying a special case of noncommutative geometry. The starting data are a supermanifold $\mathcal{M}$, its flat spin connection and complex valued $n \times n$-matrix functions defined on it. With these tools we define the noncommutative manifold we are about to study in the form of a K-cycle $(\mathcal{H}, \mathcal{D}, \Gamma)$ over the algebra $\mathcal{A}$. We will also consider the trivial fiber bundle $\mathcal{E} = \mathcal{A}$ on this noncommutative manifold.

Let us start with the algebra of matrix-valued functions on the given supermanifold

$$\mathcal{A}_0 = \Big[\mathrm{Mat}(m, \mathbf{C}) \oplus \mathrm{Mat}(m', \mathbf{C})\Big] \otimes C^\infty(\mathcal{M}, \mathcal{G}), \tag{28}$$

where $m, m' \in \mathbf{N}$, $\mathcal{M}$ is a Parisi–Sourlas supermanifold with local coordinates

$$(x^1, \cdots, x^n; \eta^1, \eta^2) \in \mathcal{M}, \tag{29}$$

and $\mathcal{G}$ is the (local) Grassmann algebra generated by η^1 and η^2. The Grassmann grading operator that commutes with bosonic numbers and anticommutes with fermionic numbers is denoted by γ_g. This algebra can be geometrically interpreted as the matrix-valued function algebra of a *two-leaf supermanifold*, if one reads off the value of the function on the first leaf from the left side of the direct sum mark and the value on the second leaf from the right side.

Only a subalgebra $\mathcal{A} \subset \mathcal{A}_0$ of this general case is used here: First, the two leaves of the manifold are associated with each other in the sense that the value of all allowed functions is the same in corresponding points on the two copies. As a consequence of this operation we get $m = m'$ and a complete symmetry in indices that distinguish between the two manifolds. Secondly, let us restrict the calculations to concern only the γ_g-even elements. The first restriction can be physically understood in such a way that it is impossible to say on which leaf events take place even if it is still possible to observe the existence of the gap between the leaves. The second restriction is imposed to obtain the right grading for the naturally arising connection fields.

The supermanifold $\mathcal{M}$ is locally a direct product of a bosonic manifold with local coordinates $(x^1, \cdots, x^n) \in \mathcal{M}_b$ and a fermionic manifold $(\eta^1, \eta^2) \in \mathcal{M}_f$ whose metrics are (g_{ij}) and (g_{IJ}) respectively[1] [8, 9]. The bosonic part $\mathcal{M}_b$ is supposed to have a spin^c-structure, so that one can consider spinors on it. The metric g_{ij} has an euclidean signature. The metric on $\mathcal{M}_f$ is

$$(g_{IJ}) = \begin{pmatrix} 0 & i\Delta \\ -i\Delta & 0 \end{pmatrix}, \qquad \Delta \in \mathbf{R}, \tag{30}$$

so that the bilinear form $X \cdot X = X^\mu X^\nu g_{\mu\nu}$ is real.

The elements on the tangent bundle $\mathrm{d}/\mathrm{d}t \in T_p\mathcal{M}$ are of the form

$$\frac{\mathrm{d}}{\mathrm{d}t} = X^i \partial_i + X^I \partial_I, \tag{31}$$

where the anticommutation relation $\{\gamma_g, X^I\} = 0$ applies in order to keep the tangent vectors bosonic. Using the tangent bundle one can define the super Clifford algebra and the Clifford bundle $\mathrm{Cliff}(T\mathcal{M}, g)$ by imposing the condition

$$\{X, Y\} = -2X \cdot Y \tag{32}$$

in the tensor algebra of $T\mathcal{M} \ni X, Y$. The bosonic part yields

$$\{\mathbf{e}^i, \mathbf{e}^j\} = -2g^{ij}, \tag{33}$$

where $\{\mathbf{e}^\mu\}$ is a local basis of the tangent space. The bosonic part of the Clifford algebra has an irreducible, antihermitian representation. The representation matrices are denoted here γ^i. The corresponding chirality operator is denoted γ_c. As the components of the tangent vectors are γ_g-odd so must the corresponding Clifford matrices

[1] The lower case Latin indices refer to coordinates on $\mathcal{M}_b$ and the upper case indices to those on $\mathcal{M}_f$. Greek indices are used when no distinction is made.

be. Thus the matrices $\mathbf{e}^I$ anticommute with the coordinates η^I and the components X^I. As a consequence of 32 they thus satisfy the algebra

$$[\mathbf{e}^I, \mathbf{e}^J] = -2g^{IJ}. \tag{34}$$

The representation matrices are denoted by

$$\Gamma^I := \frac{1}{\sqrt{\Delta}} \gamma_c \otimes \gamma_g \otimes a^I, \tag{35}$$

where the antihermitian matrices a^I satisfy the algebra

$$[a^1, a^2] = -2i \ \mathbf{1}_{\mathcal{F}}. \tag{36}$$

The letter $\mathcal{F}$ stands for the infinite dimensional representation space (Fock space) of matrices a^I. In general, the traces in this space are infinite and must thus be regularized, but it turns out that the only trace we need to calculate is that of the unity operator. This we normalize to one:

$$\mathrm{tr}\, \mathbf{1}_{\mathcal{F}} = 1 \tag{37}$$

In graded commutators we need the symbol m_μ which is defined

$$m_\mu := \begin{cases} 0, \text{if } \mu = 1, \cdots, n \\ 1, \text{if } \mu = \eta^1, \eta^2 \end{cases}. \tag{38}$$

Define further $(-)^\mu := (-1)^{m_\mu}$, $(-)^{\mu\nu} := (-1)^{m_\mu m_\nu}$ and

$$\{A^\mu, B^\nu] := A^\mu B^\nu + (-)^{\mu\nu} B^\nu A^\mu. \tag{39}$$

Using this notation the antihermitian representation matrices of the generators of the Clifford superalgebra $\mathrm{Cliff}(T\mathcal{M}, g)$ satisfy the relation

$$\{\ \Gamma^\mu, \Gamma^\nu\ ] = -2\ g^{\mu\nu}, \tag{40}$$

where $\Gamma^i := \gamma^i \otimes \mathbf{1}_{\mathcal{G}} \otimes \mathbf{1}_{\mathcal{F}}$. The nilpotent and hermitian grading operator is $\gamma_c \otimes \mathbf{1}_{\mathcal{G}} \otimes \mathbf{1}_{\mathcal{F}}$.

To show that the definitions given above are self consistent, we give an explicit realization of the Grassmann algebra. For example, the matrices $\not{n}^1$ and $\not{n}^2$ of the Clifford algebra $\mathrm{Cliff}(\mathbf{R}^4, \mathrm{diag}(-1,-1,1,1))$ generate the Grassmann algebra, that is

$$\{\not{n}^I, \not{n}^J\} = 0, \tag{41}$$

if we choose $n^1 = (1,0,1,0)$, $n^2 = (0,1,0,1)$. This Clifford algebra has a real, antihermitian representation on the exterior algebra of $\mathbf{R}^4$. Integration and derivation can be represented as matrix operations. The trace $\int \mathrm{d}^2\eta$ is normalized such that

$$\int \mathrm{d}^2\eta\ \eta^I \eta_I := \sqrt{|\ \det(g_{IJ})\ |}\ . \tag{42}$$

Integrals over $\mathbf{1}$ and η^I disappear. The grading operator γ_g is here the chirality operator. The reason for choosing the metric $\mathrm{diag}(-1,-1,1,1)$ is the fact that this is a simple way to make the Grassmann algebra close under adjunction.

Let us next define the K-cycle $(\mathcal{H}, \mathcal{D}, \Gamma)$. The Hilbert space here is

$$\mathcal{H} := S(T\mathcal{M}_b) \otimes \mathcal{G} \otimes \mathcal{F} \otimes \mathbf{C}^{2m+2k}, \tag{43}$$

where $k \in \mathbf{N}$ and $S(T\mathcal{M}_b)$ is the spinor bundle on $\mathcal{M}_b$. In this space the operator $\partial\!\!\!/ = \Gamma^\mu \partial_\mu$ is a hermitian operator and it satisfies the (ungraded) Leibnitz rule, when the inner product of $\mathcal{H}$ is

$$\langle \Psi, \Phi \rangle := \int \mathrm{d}^n x \mathrm{d}^2 \eta \, \sqrt{|\,\mathrm{sdet}\, g\,|} \; \mathrm{Tr}(\Psi^* \Phi). \tag{44}$$

The Dirac operator $\mathcal{D}$ can be defined in several ways. The simplest definition is

$$\mathcal{D} = \begin{pmatrix} \partial\!\!\!/ \otimes \mathbf{1}_m \otimes \mathbf{1}_k & \Gamma \otimes M \otimes K \\ \Gamma \otimes M \otimes K & \partial\!\!\!/ \otimes \mathbf{1}_m \otimes \mathbf{1}_k \end{pmatrix}, \tag{45}$$

where the grading is $\Gamma = \sigma^3 \otimes \gamma_c \otimes \gamma_g$ and where M and K are some given constant hermitian matrices. In the Standard Model [2] M is a fermion mass matrix and K is the Kobayashi–Maskawa matrix that mixes fermion families with each other. Matrix M operates in the same space as the elements of $\mathcal{A}$. In the present case, K operates in $\mathbf{C}^k$ and it does not have common indices with any other object. We would have been able to use the whole spin connection $\partial\!\!\!/ + \omega\!\!\!/$, but since ω disappears from all calculations, we omit this unnecessary complication. This definition produces, as we shall presently see, the usual Parisi–Sourlas Yang–Mills theory.

Because the calculation is not restricted by summability assumptions, one can as well study the operator

$$\mathcal{D} := \begin{pmatrix} \partial\!\!\!/ \otimes \mathbf{1}_m \otimes \mathbf{1}_k & \gamma_c \otimes M \otimes K \\ \gamma_c \otimes M \otimes K & \partial\!\!\!/ \otimes \mathbf{1}_m \otimes \mathbf{1}_k \end{pmatrix}, \tag{46}$$

which gives rise to a different classical field theory. Furthermore, one can scale the components of $\mathcal{D}$ by any (γ_g-even) function, here denoted χ and ξ. This means considering operator

$$\mathcal{D} = \begin{pmatrix} \xi \, \partial\!\!\!/ \otimes \mathbf{1}_m \otimes \mathbf{1}_k & \chi \, \gamma_c \otimes M \otimes K \\ \chi \, \gamma_c \otimes M \otimes K & \xi \, \partial\!\!\!/ \otimes \mathbf{1}_m \otimes \mathbf{1}_k \end{pmatrix}. \tag{47}$$

We assume definition 46 and comment later on the other two possibilities.

5 The super Yang–Mills theory of $\mathcal{M}$

In this section the action 14 is calculated for the NCG defined in the previous section and expressed as a functional of physical fields. The procedure relies on the methods that Chamseddine *et al.* presented in [3]. Let

$$\alpha = \sum (a^n \otimes \mathbf{1}_2) \mathrm{d}(b^n \otimes \mathbf{1}_2) \in \Omega^1 \mathcal{A} \tag{48}$$

$$a^n, b^n \in \mathrm{Mat}(m, \mathbf{C}) \otimes C^\infty(\mathcal{M}, \mathcal{G}), \quad \sum a^n b^n = \mathbf{1}. \tag{49}$$

In what follows, tensor products and unity operators are omitted where there is no danger of confusion. We will write the curvature $\pi(\mathrm{d}\alpha + \alpha^2)$ down with the help of free fields some of which turn out to be nondynamical and which do not appear in $\pi(\alpha)$. These auxiliary fields are eliminated with the help of appropriate Euler–Lagrange equations so that the action $S[\alpha]$ is in its minimum for a given $\pi(\alpha)$.

First, the dynamical fields are

$$\pi(\alpha) = \sum_n a^n \; i[\mathcal{D}, b^n] = i \begin{pmatrix} \not{A} & \gamma_c KH \\ \gamma_c KH & \not{A} \end{pmatrix}, \tag{50}$$

where the notation

$$\begin{aligned} A_\mu &:= \sum a^n \partial_\mu b^n \quad \text{and} && (51) \\ H &:= \sum a^n [M, b^n] && (52) \end{aligned}$$

is used. The gauge transformations of these fields are

$$\gamma_u(A_\mu) = u A_\mu u^* + u \partial_\mu u^* \tag{53}$$
$$\gamma_u(H + M) = u(H + M)u^*, \tag{54}$$

where $u \in \mathrm{U}(\mathrm{Mat}(m, \mathbf{C}) \otimes C^\infty(\mathcal{M}, \mathcal{G}))$. Thus the field A_μ transforms like a gauge field and $H + M$ like a scalar field in the adjoint representation of the gauge group.

Due to the fact that the elements of $\mathcal{A}$ are γ_g-even, the components of the fields A^μ satisfy the grading condition

$$[A^\mu_{ab}, A^\nu_{cd}\} = 0 \tag{55}$$

and A can be interpreted as the Parisi–Sourlas gauge field. It transforms in the transformations of the orthosymplectic group $\mathrm{OSp}(n|2)$ as a vector so that $\mathrm{tr}(A^\mu A_\mu)$ is invariant.

Because π is a homomorphism, one obtains

$$\pi(\alpha^2) = - \begin{pmatrix} \not{A}^2 + K^2 H^2 & [\not{A}, H\} K \gamma_c \\ [\not{A}, H\} K \gamma_c & \not{A}^2 + K^2 H^2 \end{pmatrix} \tag{56}$$

and

$$\pi(\mathrm{d}\alpha) = - \begin{pmatrix} \not{\partial}\not{A} + Z + K^2\big(\{M, H\} - Y\big) & \big([\not{A}, M\} - \not{P} + \not{\partial} H\big) K \gamma_c \\ \big([\not{A}, M\} - \not{P} + \not{\partial} H\big) K \gamma_c & \not{\partial}\not{A} + Z + K^2\big(\{M, H\} - Y\big) \end{pmatrix}, \tag{57}$$

where

$$\begin{aligned} Z &:= \sum a^n \partial^\mu \partial_\mu b^n && (58) \\ Y &:= \sum a^n [M^2, b^n] && (59) \\ \not{P} &:= 2 \sum a^n M \Gamma^I \partial_I b^n. && (60) \end{aligned}$$

The fields Z, Y and $\not{P}$ are thus the naturally appearing auxiliary fields. The curvature is now

$$\pi(\vartheta) = -\begin{pmatrix} \not{\partial}\not{A} + \not{A}^2 + Z + K^2\Big(\Phi^2 - \Psi\Big) & \not{\partial}\Phi + [\not{A}, \Phi\} - \not{P} \\ \not{\partial}\Phi + [\not{A}, \Phi\} - \not{P} & \not{\partial}\not{A} + \not{A}^2 + Z + K^2\Big(\Phi^2 - \Psi\Big) \end{pmatrix}, \tag{61}$$

where the new fields are $\Phi := H + M$ and $\Psi := Y + M^2$. If one had used the definition 45 for the Dirac operator $\mathcal{D}$, one would not have obtained the field $\not{P}$ at all, and the graded commutator in the formula 61 above would be an ordinary commutator.

The trace Tr in formula 14 can be calculated in each tensor product space separately

$$\mathrm{Tr} = \mathrm{tr}_m \otimes \mathrm{tr}_k \otimes \mathrm{tr}_{\mathcal{F}} \otimes \mathrm{tr}_{\mathrm{Cl}} \otimes \mathrm{tr}_2 \tag{62}$$

in obvious notation. All traces are normalized so that the trace of the unit operator is one. Define

$$\Sigma^{\mu\nu} := \frac{1}{4}[\Gamma^\mu, \Gamma^\nu\} \tag{63}$$

so that $\Gamma^\mu\Gamma^\nu = -g^{\mu\nu} + 2\Sigma^{\mu\nu}$, and

$$F_{\mu\nu} := \partial_\mu A_\nu - (-)^{\mu\nu}\partial_\nu A_\mu + [A_\mu, A_\nu\}. \tag{64}$$

One can use the orthogonality properties of the bosonic γ-matrices

$$\begin{aligned} \mathrm{tr}_{\mathrm{Cl}}\, \mathbf{1}^2 &= 1 && (65)\\ \mathrm{tr}_{\mathrm{Cl}}\, \gamma^i\gamma_c\gamma^j\gamma_c &= g^{ij} && (66)\\ \mathrm{tr}_{\mathrm{Cl}}\, \sigma^{ij}\sigma^{nm} &= \frac{1}{4}(g^{im}g^{jn} - g^{in}g^{jm}). && (67) \end{aligned}$$

Define also

$$\begin{aligned} \alpha^{IJ} &:= \frac{1}{4\Delta}\,\mathrm{tr}\{a^I, a^J\} && (68)\\ \alpha^{IJKL} &:= \frac{1}{16\Delta^2}\Big(\mathrm{tr}(\{a^I, a^J\}\{a^K, a^L\}) - \mathrm{tr}\{a^I, a^J\}\,\mathrm{tr}\{a^K, a^L\}\Big). && (69) \end{aligned}$$

Note that $\mathrm{tr}_{\mathcal{F}}(a^I a^J)\,\mathrm{tr}_m\, C_I C_J = \Delta\,\mathrm{tr}_m\, C^I C_I$. In above notation this yields

$$\begin{aligned} \mathcal{L} = &-\frac{1}{2}\,\mathrm{tr}\, F^{\mu\nu}F_{\mu\nu} + (\alpha^{IJKL} + \frac{1}{2}g^{KI}g^{LJ})F_{IJ}F_{KL} + \\ &+ \mathrm{tr}\, K^2\,\mathrm{tr}(\nabla\Phi + [\mathbf{A}, \Phi])^2 + (\mathrm{tr}\, K^4 - \mathrm{tr}^2\, K^2)\,\mathrm{tr}(\Phi^2 - \Psi)^2 + \\ &+ \mathrm{tr}\Big(- \alpha^{IJ}F_{IJ} - (-)^\mu\partial^\mu A_\mu - (-)^\mu A^\mu A_\mu + Z + \mathrm{tr}\, K^2(\Phi^2 - \Psi)^2\Big) + \\ &- \mathrm{tr}\Big(\partial^I\Phi + \{A^I, \Phi\} - P^I\Big)\Big(\partial_I\Phi + \{A_I, \Phi\} - P_I\Big), \end{aligned} \tag{70}$$

where $\nabla^i := \partial^i$ and $\mathcal{L} := \mathrm{Tr}\,\pi(\vartheta)^2$.

The next step is to eliminate the auxiliary fields using their Euler–Lagrange equations. Obviously, trying to eliminate the field Ψ would kill the potential of the scalar

field. The only way to save the potential is to constrain the matrices M to be hermitian matrices with the property

$$M^2 = \frac{1}{2}\mu^2\mathbf{1} + \beta M \tag{71}$$

for some real μ and β, because this yields $\Psi = (1/2)\ \mu^2\mathbf{1} + \beta\Phi$. The two equations of motion to be satisfied are

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial\partial_\mu Z_{ab}} - \frac{\partial \mathcal{L}}{\partial Z_{ab}} = 0 \tag{72}$$

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial\partial_\mu P^I_{ab}} + \frac{\partial \mathcal{L}}{\partial P^I_{ab}} = 0 \tag{73}$$

for all I and (a, b). The resulting constraints are

$$0 = \partial^I\Phi + \{A^I, \Phi\} - P^I \tag{74}$$

$$0 = -\alpha^{IJ}F_{IJ} - (-)^\mu\partial^\mu A_\mu - (-)^\mu A^\mu A_\mu + Z + \operatorname{tr} K^2(\Phi^2 - \Psi). \tag{75}$$

Substituting equations (74The super Yang–Mills theory of $\mathcal{M}$equation.74 – 75The super Yang–Mills theory of $\mathcal{M}$equation.74) to the lagrangian density 70 one obtains

$$\begin{aligned}\mathcal{S}_{YM} = &\int \mathrm{d}^n x \mathrm{d}^2\eta \sqrt{|\operatorname{sdet} g\,|}\,\Big(-\frac{1}{2}\operatorname{tr}\Big(F^{\mu\nu}F_{\mu\nu}\Big) + \\ &+ \operatorname{tr}(K^2)\operatorname{tr}\Big((D^i\Phi)^* D_i\Phi\Big) + \Big(\operatorname{tr} K^4 - \operatorname{tr}^2 K^2\Big)\operatorname{tr} V(\Phi)\Big),\end{aligned} \tag{76}$$

where the potential is

$$V(\Phi) := \Phi^4 - 2\beta\Phi^3 + (\beta^2 - \mu^2)\Phi^2 + \beta\mu^2\Phi \tag{77}$$

and the covariant derivative is

$$D^\mu := \partial^\mu + [A^\mu,\quad]. \tag{78}$$

Formula 76 resulted from the the fact that the second term in 70 disappears identically. This happens, if we demand that the elements of $\mathcal{A}$ are invariant under the transformations of the orthosymplectic group $\mathrm{OSp}(n|2)$ and no (other) Grassmann parameters appear in $\mathcal{A}$. Then the elements are of the form

$$a^n(x^\mu) = a^n_0(x^i x_i) + a^n_2(x^i x_i)\eta^I\eta_I \in \mathcal{A} \tag{79}$$

and a part of the components of the field tensor disappear, namely $F_{IJ} = 0$. If one can choose such a regularization that the coefficients

$$\alpha^{IJKL} + \frac{1}{2}g^{KI}g^{LJ} \tag{80}$$

vanish the same result 76 is obtained.

Action 76 is clearly positive definite, because $\operatorname{tr} K^4 \geq \operatorname{tr}^2 K^2$ holds. proportional to the identity. The fields A_μ are antihermitian and the field Φ is hermitian. The action is obviously gauge invariant. In this model the scalar fields do not propagate to the fermionic direction. On the classical level it is impossible to choose the parameters so that the self-interaction would disappear and the scalar fields stay massive.

If the Dirac operator had been defined by formula 45 instead of the formula 46 the resulting action would have been

$$\begin{aligned} S_{YM} &= \int \mathrm{d}^n x \mathrm{d}^2\eta \, \sqrt{|\operatorname{sdet} g\,|} \, \Big(-\frac{1}{2} \operatorname{tr} \Big(F^{\mu\nu} F_{\mu\nu} \Big) + \\ &\quad + \operatorname{tr}(K^2) \operatorname{tr} \Big((D^\mu \Phi)^* D_\mu \Phi \Big) + \Big(\operatorname{tr} K^4 - \operatorname{tr}^2 K^2 \Big) \operatorname{tr} V(\Phi) \Big), \end{aligned} \tag{81}$$

which is the conventional super Yang–Mills theory coupled to a scalar field.

6 The induced QCD

Kazakov and Migdal have studied [11] the induced QCD starting from the lagrangian density

$$\mathcal{L}_{KM} = \frac{N}{g^2} \operatorname{tr} \Big((\nabla \Phi + ig\,[\mathbf{A}, \Phi])^2 + m^2 \Phi^2 + \lambda_0 \Phi^4 \Big). \tag{82}$$

The idea is that after the path integration over the scalar field in the generating functional

$$\mathcal{W}[J^i] = \int [\mathrm{d}\Phi][\mathrm{d}A] \exp \Big(- \int \mathrm{d}^n x \, \sqrt{|\det g_{ij}\,|} \, (\mathcal{L}_{KM} - J^i A_i) \Big) \tag{83}$$

the resulting effective theory would be QCD. In this section the possibility of obtaining this lagrangian $\mathcal{L}_{KM}$ from the noncommutative geometry of some space is studied.

In the previous section a Yang–Mills theory was derived from a suitable noncommutative geometry. The most general deformation of the conventional Yang–Mills theory is obtained by deforming the Dirac operator as in formula 47. A similar procedure presented by Chamseddine *et al.* in [5] gives rise to a theory that involves a scalar field in Einstein–Hilbert gravitation theory.

Let us now assume the definition 47 for the Dirac operator. All the calculations of the previous sections go through so that one only needs to scale derivatives in the resulting expression by ξ and K-matrices by χ. The action thus obtained is

$$\begin{aligned} \mathcal{S}_{YM} &= \int \mathrm{d}^n x \mathrm{d}^2\eta \, \sqrt{|\operatorname{sdet} g\,|} \, \Big(-\frac{1}{2} \xi^4 \operatorname{tr} \Big(F^{\mu\nu} F_{\mu\nu} \Big) + \\ &\quad + \xi^2 \chi^2 \operatorname{tr}(K^2) \operatorname{tr} \Big((D^i \Phi)^* D_i \Phi \Big) + \chi^4 \Big(\operatorname{tr} K^4 - \operatorname{tr}^2 K^2 \Big) \operatorname{tr} V(\Phi) \Big), \end{aligned} \tag{84}$$

Special forms for the functions ξ and χ are chosen, namely

$$\xi := \varepsilon \, e^{\eta^I \eta_I} + \mathcal{O}(\varepsilon^2) \tag{85}$$

$$\chi := \varepsilon \, e^{\eta^I \eta_I / \varepsilon^4} + \mathcal{O}(\varepsilon^2). \tag{86}$$

This yields further

$$\begin{aligned} \xi^4 &= \mathcal{O}(\varepsilon^4) && (87)\\ \xi^2\chi^2 &= 2\eta^I\eta_I + \mathcal{O}(\varepsilon^4) && (88)\\ \chi^4 &= 4\eta^I\eta_I + \mathcal{O}(\varepsilon^4). && (89) \end{aligned}$$

In the notation

$$\begin{aligned} \Phi(x^\mu) &= \Phi_0(x^i) + \Phi_2(x^i)\eta^I\eta_I && (90)\\ A^\nu(x^\mu) &= A_0^\nu(x^i) + A_2^\nu(x^i)\eta^I\eta_I && (91)\\ D_0^i &= \partial^i + [A_0^i, \ \ ] && (92) \end{aligned}$$

and after Grassmann integration we obtain the action

$$\mathcal{S}_{YM} = \int \mathrm{d}^n x \sqrt{|\det g_{ij}|}\ \mathrm{tr}\left(2\,\mathrm{tr}(K^2)\ (D_0^i\Phi_0)(D_{0i}\Phi_0) + 4(\mathrm{tr}\,K^4 - \mathrm{tr}^2\,K^2)\ V(\Phi_0)\right). \quad (93)$$

The result 93 is of the same form as the continuum limit of the induced QCD 82, when one redefines the fields $\Phi_0 \longrightarrow \Phi$ and $A^i \longrightarrow igA$. The parameters are given by formulae

$$\begin{aligned} \frac{N}{g^2} &= 2\,\mathrm{tr}\,K^2 && (94)\\ \lambda_0 &= 2\frac{\mathrm{tr}\,K^4}{\mathrm{tr}\,K^2} - 2\,\mathrm{tr}\,K^2 && (95)\\ m^2 &= \lambda_0\mu^2. && (96) \end{aligned}$$

In order to maintain the fields Φ and A_i purely bosonic in gauge transformations, the local gauge symmetry is reduced to the group $\mathrm{U}(m, \mathbf{C})$. This limit of the super Yang–Mills theory thus correctly produces the purely bosonic Kazakov–Migdal theory.

It is essential that one studies the equations of motion for some finite ε, because otherwise the constraint 75 would look different *c.f.* 16. Other choices of functions ξ and χ are possible. The outcome of calculations is essentially the same also with the choice $\xi = \xi_0\gamma_c\Gamma^I\eta_I + \kappa$ to the zeroth order in κ.

The model has some features that are absent from the Kazakov–Migdal model. The calculation is a way to show the existence of a hidden supersymmetry in the model. The interpretation of the matrix K as a Kobayashi–Maskawa matrix gives a motivation to include fermion families with nontrivial mixing. The formula 96 means that the scalar particle having mass implies it moves in a nontrivial potential, at least on the classical level. In the quantum theory this does not imply a massive particle having necessarily nontrivial self-interaction: it has been shown in the case of the Standard model that some classically predicted restrictions on the parameter values do not hold in a quantized theory [12]. Formula 96 has also an other interesting feature: namely, the parameter $1/\mu$ roughly describes the distance between the two leaves of the manifold. Thus the mass of the scalar particle is inversely proportional to this

length scale, so that the proportionality constant is the square root of the coupling constant. This gives rise to the geometric picture that when the two leaves approach each other the mass of the scalar particle tends to infinity (for a given λ), and when the distance between the leaves increases without bounds, the scalar becomes massless. The strength of the interaction is connected with the *untriviality* of the fermion mixing, or the Kobayashi–Maskawa matrix.

Considering other functions ξ and χ would naturally lead to different results. For instance, as a by-product of the calculation above it is apparent that replacing ξ by χ would naturally produce an ordinary, purely bosonic Yang–Mills theory.

Let us next find out what would have been the outcome of the analysis, if one would have considered the Dirac operator 45. This amounts to scaling the action 81 with ξ and χ in the above described manner. It yields

$$\begin{aligned}\mathcal{S}_{YM} &= \int \mathrm{d}^n x \mathrm{d}^2\eta \, \sqrt{|\, \mathrm{sdet}\, g \,|}\, \Big(-\frac{1}{2}\xi^4 \,\mathrm{tr}\, \Big(F^{\mu\nu} F_{\mu\nu} \Big) + \\ &\quad + \xi^2\chi^2 \,\mathrm{tr}(K^2)\, \mathrm{tr}\, \Big((D^\mu \Phi)^* D_\mu \Phi \Big) + \chi^4 \Big(\mathrm{tr}\, K^4 - \mathrm{tr}^2\, K^2 \Big)\, \mathrm{tr}\, V(\Phi) \Big). \end{aligned} \tag{97}$$

After Grassmann integration the resulting action differs from the action 93 only by

$$\delta\mathcal{S}[A, \Phi_0, \Phi_2] = 2\,\mathrm{tr}\, K^2 \int \mathrm{d}^n x \sqrt{|\, \det g_{ij} \,|}\;\; \mathrm{tr}(\Phi_2 + [A, \Phi_0])^2 \tag{98}$$

where the notation is $A^I := A\eta^I$. The only relevant matrix indices are those in $\mathbf{C}^m$. Since both Φ_i and $-iA$ are hermitian, we can choose as the basis of the traceless, hermitian generators T^a of $\mathrm{SU}(m)$ and the unity operator. The coefficients t^{abc} are the structure constants of $\mathrm{SU}(m)$. Let us path integrate over the field A. This can be done after a change of variables

$$B^c := \Phi_2^c - t^{abc} A^a \Phi_0^b. \tag{99}$$

The jacobian associated to the transformation is

$$\mathrm{Det} \left| \frac{\delta i A^a(y)}{\delta B^b(x)} \right| = \mathrm{Det}^{-1} \left| i t^{abc}\; \Phi_0^c(x)\; \delta(x-y) \right| = \mathrm{Det}^{-1} \left| \Phi_0 \right| = \exp\Big(-\int \mathrm{tr} \ln \Phi_0 \Big). \tag{100}$$

It does not depend on B so the Gaussian integration can be performed. One can also easily integrate over Φ_2. The effective contribution to the action is thus

$$\delta\mathcal{S}_{\mathrm{eff}}[\Phi_0] = \int \mathrm{tr} \ln \Phi_0. \tag{101}$$

The logarithmic term in the potential introduces new interactions, and the theory is not renormalizable. Thus the only way to produce a consistent bosonic theory with these tools starting with a superspace would seem to be to consider the Dirac operator 46.

7 Conclusions

The studied NCG describes a two–leaf supermanifold. The two leaves are identified, so that it is not possible to detect on which leaf physical events take place. On this manifold a noncommutative fiber bundle structure has been studied. The bundle structure has been written in terms of ordinary principal bundles and with the help of the associated connection field A^μ. Connes' formalism introduces also a scalar field Φ that appeared to be a connection field into the discrete direction. The mass of the scalar fields give the order of the magnitude of the distance between the leaves. The strength of the self interaction of the scalars depends only on the fermion structure through the K-matrix.

There has remained some freedom to choose the Dirac operator in the K-cycle. This freedom has been investigated and some of its consequences in the quantum field theory have been pointed out. The curvature of a two-leaf Parisi-Sourlas supermanifold has been shown to define a super Yang–Mills theory in a similar way as it does in a purely bosonic case. This Yang–Mills theory had the property that it could be continuously deformed into a purely bosonic action for the induced QCD in the Kazakov–Migdal model. The set of field theories that can be formulated in the formalism of Connes' NCG has thus been extended to include also concrete theories on supermanifolds. The treatment is, however, classical. The quantization of field theories in NCG remains an open problem.

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